Interpolating Wavelet Galerkin Model of Time Dependent Inhomogeneous Electrically-Large Optical Waveguide Problems

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Abstract—**Biorthogonal interpolating wavelets have been applied to inhomogeneous electromagnetic field modeling through the wavelet-Galerkin scheme, yielding a simple and versatile algorithm for the time dependent Maxwell's equations. The resulting scheme significantly reduces the computational expenditure particularly in the modeling of electrically large optical waveguides while maintaining high accuracy.**

Keywords—**Time domain electromagnetic field analysis, inhomogeneous media, biorthogonal wavelets, Deslauriers-Dubuc interpolating basis**

I. Introduction

Wavelets have been widely investigated to improve the efficiency in solving partial differential equations. Battle-Lemarié wavelets have been applied to solve Maxwell's equations in time domain in [1], which enables numerical analysis with less computational expenditure due to its highly linear dispersion characteristics compared to the conventional space discrete methods. The shifted interpolation property of the Daubechies compactly supported scaling function has been utilized in [2] to accommodate local sampling of the field despite the large support of the basis. The interpolation property of the basis functions significantly simplifies the numerical procedures and increases the versatility of the numerical analysis. The authors have extended the scheme by using Daubechies scaling functions with higher regularity [3] and have shown that the extended scheme has an advantage in analyzing electrically large inhomogeneous structures both for two-dimensions [4], [5], and three-dimensions [6].

Recently, the authors developed a more efficient higher-order scheme [7] using wavelets of exact interpolation [8], [9]; the Deslauriers-Dubuc interpolating functions [10], [11] are adopted as the fundamental scaling basis and their shifted and contracted versions as the additional wavelet basis. These functions constitute non-L^2 biorthogonal bases that are smooth, symmetric, compactly supported and exactly interpolating. Unlike the Daubechies orthogonal wavelets [12], of which interpolation property is limited to the bases of low regularity [3], the proposed basis set yields a scheme of desired order of regularity as well as saves the computational overhead for the total field reconstruction.

This paper further extends the interpolating wavelet scheme to arbitrary inhomogeneous media. By virtue of the sophisticated nature of the basis functions, the extension retains the simplicity of the numerical procedure while significantly increasing the versatility. Furthermore, by applying wavelets only where higher resolution is desired, the proposed scheme provides a sub-gridding capability. Verification is performed by the analysis of two-dimensional dielectric waveguides that have a typical dimension of widely used optical waveguides.

II. Formulation

We adopt the Deslauriers-Dubuc interpolating function ϕ of order $2p-1$ as a scaling function, which satisfies the so-called dilation relation [13]

$$\phi(x) = \sum_{k=-\infty}^{+\infty} h_k^* \phi(2t-k) \; , \tag{1}$$

where the filter coefficient h_k^* in is obtained from the filter of the Daubechies compactly supported wavelets of p-vanishing moments h_k [12] by

$$h_k^* = \sum_{m=-\infty}^{+\infty} h_m h_{m-k} \; . \tag{2}$$

For $p = 2$, $\{h_k^* | k = -3, -2, \cdots, 3\} = \{-0.0625$, 0, 0.5625, 1.0, 0.5625, 0, $-0.0625\}$ and for $p = 4$, $\{h_k^* | k = -7, -6, \cdots, 7\} = \{-0.00244141$, 0, 0.0239258, 0, -0.119629, 0, 0.598145, 1.0, 0.598145, 0, -0.119629, 0, 0.0239258, 0, $-0.00244141\}$. Figure 1 shows ϕ for $p = 2, 4$ and 10. This function has even symmetry and minimum support of $[-2p+1, 2p-1]$ to represent a polynomial of degree $(2p-1)$.

Then, the wavelet function which creates the 'detail' space can be chosen as

$$\psi(x) = \phi(2x-1) \; . \tag{3}$$

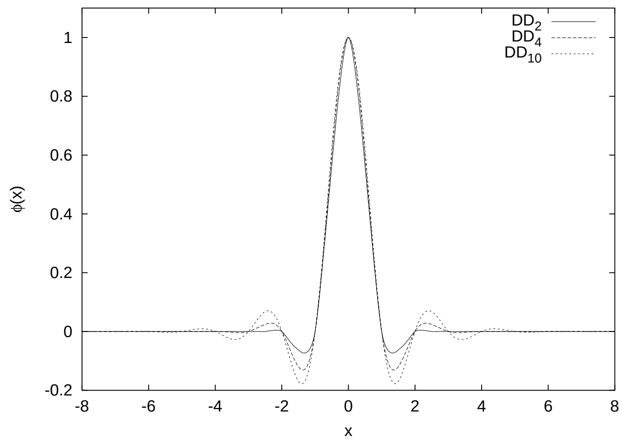

Fig. 1. Deslauriers-Dubuc interpolating function of order $2p-1$ for $p = 2, 4$ and 10.

Although (3) is not a true wavelet since it has no vanishing moment, it generates multiresolution analysis and plays the same role as other wavelets in the context of the Galerkin procedure. The dual functions can be chosen as the Dirac delta function and a linear combination of those as

$$\tilde{\phi}(x) = \delta(x) , \tag{4}$$

$$\tilde{\psi}(x) = \sum_{k=-2p+2}^{2p} (-1)^{k-1} h^*_{-k+1} \delta(x - \frac{k}{2}) . \tag{5}$$

Since the Dirac delta is not in the space of square integrable functions, the resulting basis set is in non-L^2 space. Let $f_j(x) = f((x/\Delta x) - j)$ for $f = \phi, \psi, \tilde{\phi}$ and $\tilde{\psi}$ with Δx being the spatial discretization interval, then the set of the basis functions satisfies the biorthogonal relations

$$\begin{aligned} < \phi_i, \tilde{\phi}_j > &= \delta_{ij} , \quad < \psi_i, \tilde{\psi}_j > = \delta_{ij} , \\ < \phi_i, \tilde{\psi}_j > &= < \psi_i, \tilde{\phi}_j > = 0 . \end{aligned} \tag{6}$$

We consider Maxwell's curl equations for inhomogeneous isotropic lossy media in the 2D TE case where a wave propagates in the xz-plane,

$$-\mu \frac{\partial H_x}{\partial t} = -\frac{\partial E_y}{\partial z} , \tag{7}$$

$$-\mu \frac{\partial H_z}{\partial t} = \frac{\partial E_y}{\partial x} , \tag{8}$$

$$J_y + \sigma \frac{D_y}{\varepsilon} + \frac{\partial D_y}{\partial t} = \frac{\partial H_x}{\partial z} - \frac{\partial H_z}{\partial x} , \tag{9}$$

and the constitutive equation

$$D_y = \varepsilon E_y . \tag{10}$$

The electromagnetic field is first expanded in the scaling function ϕ (1) and the wavelet function ψ (3) in space and in the Haar scaling functions $h(t)$ [1] in time as

$$\begin{aligned} E_y(x,z,t) = \sum_{i,k,n=-\infty}^{+\infty} \{ & E^{y,\phi\phi}_{i,k,n+1/2} \, \phi_i(x) \, \phi_k(z) \\ & + E^{y,\phi\psi}_{i,k+1/2,n+1/2} \, \phi_i(x) \, \psi_k(z) \\ & + E^{y,\psi\phi}_{i+1/2,k,n+1/2} \, \psi_i(x) \, \phi_k(z) \\ & + E^{y,\psi\psi}_{i+1/2,k+1/2,n+1/2} \, \psi_i(x) \, \psi_k(z) \} \, h_{n+1/2}(t) , \end{aligned} \tag{11}$$

$$\begin{aligned} H_x(x,z,t) = \sum_{i,k,n=-\infty}^{+\infty} \{ & H^{y,\phi\phi}_{i,k+1/2,n} \, \phi_i(x) \, \phi_{k+1/2}(z) \\ & + H^{y,\phi\psi}_{i,k+1,n} \, \phi_i(x) \, \psi_{k+1/2}(z) \\ & + H^{y,\psi\phi}_{i+1/2,k+1/2,n} \, \psi_i(x) \, \phi_{k+1/2}(z) \\ & + H^{y,\psi\psi}_{i+1/2,k+1,n} \, \psi_i(x) \, \psi_{k+1/2}(z) \} \, h_n(t) , \end{aligned} \tag{12}$$

and similarly for D_y and H_z, where $h_n(t) = h(\frac{t}{\Delta t} - n + \frac{1}{2})$ and Δt being the time step. The expansion coefficients for the wavelet terms are defined on the Yee cell [14] at nodes halfway between the regular nodes in the direction of the corresponding wavelets as in [1]. As in the Galerkin procedure, the field expansions are substituted in the equations and they are tested with the dual functions (4) and (5). The constitutive equation (10) is discretized as

$$D^{y,\phi\phi}_{i,k,n+1/2} = \varepsilon_{i,k} E^{y,\phi\phi}_{i,k,n+1/2} , \tag{13}$$

$$\begin{aligned} D^{y,\phi\psi}_{i,k+1/2,n+1/2} &= \sum_l \epsilon^{\phi\psi}_{i,k}(l) E^{y,\phi\phi}_{i,k+l,n+1/2} \\ &+ \varepsilon_{i,k+1/2} E^{y,\phi\psi}_{i,k+1/2,n+1/2} , \end{aligned} \tag{14}$$

and the other wavelet terms $D^{y,\psi\phi}$ and $D^{y,\psi\psi}$ are obtained similarly. In (13) and (14), ε denotes the local dielectric constant defined by $\varepsilon_{i,k} \equiv \varepsilon(i\Delta x, k\Delta z)$, and $\epsilon^{\phi\psi}_{i,k}(l)$ denotes non-zero inner products between ϕ and $\tilde{\psi}$ due to the inhomogeneity of the media, i.e.,

$$\begin{aligned} \epsilon^{\phi\psi}_{i,k}(l) &\equiv \left\langle \phi_k(z)\varepsilon(x,z) \,\middle|\, \tilde{\psi}_{k-l}(z) \right\rangle \\ &= h^*_{-2l+1}(\varepsilon_{i,k-l+1/2} - \varepsilon_{i,k}) . \end{aligned} \tag{15}$$

An important observation here is that (13) has a local relation that does *not* require other wavelet terms for update while (14) involves $E^{y,\phi\phi}$. Substituting (13) and (14) into the discretized equations of (9), we eliminate the flux density D_y from the final time evolution equations as

$$E^{y,\phi\phi}_{i,k,n+1/2} = \frac{2\epsilon - \sigma\Delta t}{2\epsilon + \sigma\Delta t} E^{y,\phi\phi}_{i,k,n-1/2} + \frac{2\Delta t}{2\epsilon + \sigma\Delta t} \left\{ \frac{1}{\Delta z} \left[\sum_l a_{\phi\phi}(l) H^{x,\phi\phi}_{i,k+l+1/2,n} + \sum_l a_{\psi\phi}(l) H^{x,\phi\psi}_{i,k+l+1,n} \right] - \frac{1}{\Delta x} \left[\sum_l a_{\phi\phi}(l) H^{z,\phi\phi}_{i+l+1/2,k,n} + \sum_l a_{\psi\phi}(l) H^{z,\phi\psi}_{i+l+1/2,k+1/2,n} \right] - J^{y,\phi\phi}_{i,k,n} \right\}, \quad (16)$$

$$E^{y,\phi\psi}_{i,k+1/2,n+1/2} = \frac{2\epsilon - \sigma\Delta t}{2\epsilon + \sigma\Delta t} (E^{y,\phi\psi}_{i,k+1/2,n-1/2} + F^{y,\phi\psi}_{i,k+1/2,n-1/2}) - F^{y,\phi\psi}_{i,k+1/2,n+1/2} + \frac{2\Delta t}{2\epsilon + \sigma\Delta t} \left\{ \frac{1}{\Delta z} \left[\sum_l a_{\phi\psi}(l) H^{x,\phi\phi}_{i,k+l+1/2,n} + \sum_l a_{\psi\psi}(l) H^{x,\phi\psi}_{i,k+l+1,n} \right] - \frac{1}{\Delta x} \left[\sum_l a_{\phi\phi}(l) H^{z,\phi\psi}_{i+l+1/2,k+1/2,n} + \sum_l a_{\psi\phi}(l) H^{z,\psi\psi}_{i+l+1,k+1/2,n} \right] - J^{y,\phi\psi}_{i,k+1/2,n} \right\}, \quad (17)$$

where

$$F^{y,\phi\psi}_{i,k+1/2,n+1/2} = \sum_{l=-p+1}^{p} \frac{\epsilon^{\phi\psi}_{i,k}(l)}{\varepsilon_{i,k+1/2}} E^{y,\phi\phi}_{i,k+l,n+1/2} \quad (18)$$

is the inhomogeneity correction term that is non-zero in the vicinity of dielectric interface over $2p-1$ cells and vanishes in a homogeneous region; the summation is taken for $l = [-p+1, p]$ with p being the index of the scaling function. The other field components of $E^{y,\psi\phi}$, $E^{y,\psi\psi}$, and those for H_x and H_z can be obtained similarly. This formulation is highly versatile; the scaling function coefficients $E^{y,\phi\phi}$ are first updated throughout the analysis region, then the wavelet coefficients $E^{y,\phi\psi}$ can be updated using the knowledge of the scaling function coefficients. Although additional memory is required to store F for the previous time step instead of D_y, the wavelet terms can be applied locally at limited regions, resulting in a sub-gridding algorithm.

This procedure serves to avoid matrix inversion that is required for other wavelet families such as Battle-Lemarié, B-spline, Daubechies orthogonal, and Cohen-Daubechies-Feauveau biorthogonal wavelets when it comes to a inhomogeneous region. The summation with respect to the stencil l in (16) and (17) is taken according to the number of the connection coefficients $a_{\xi\zeta}$ for $\xi, \zeta = \phi, \psi$. The connection coefficients $a_{\xi\zeta}(l)$ are given by closed form expressions

$$a_{\phi\phi}(l) \equiv \left\langle \frac{d\phi_{i+1/2}}{dx} \middle| \tilde{\phi}_{i-l} \right\rangle = \left. \frac{d\phi(x)}{dx} \right|_{x=-l-\frac{1}{2}}, \quad (19)$$

$$a_{\phi\psi}(l) \equiv \left\langle \frac{d\phi_{i+1/2}}{dx} \middle| \tilde{\psi}_{i-l} \right\rangle = \sum_{k=-2p+2}^{2p} (-1)^{k-1} h^*_{-k+1} \left. \frac{d\phi(x)}{dx} \right|_{x=-l+\frac{k}{2}-\frac{1}{2}}, \quad (20)$$

$$a_{\psi\phi}(l) \equiv \left\langle \frac{d\psi_{i+1/2}}{dx} \middle| \tilde{\phi}_{i-l} \right\rangle = 2 \left. \frac{d\phi(x)}{dx} \right|_{x=-2l-2}, \quad (21)$$

$$a_{\psi\psi}(l) \equiv \left\langle \frac{d\psi_{i+1/2}}{dx} \middle| \tilde{\psi}_{i-l} \right\rangle = 2 \sum_{k=-2p+2}^{2p} (-1)^{k-1} h^*_{-k+1} \left. \frac{d\phi(x)}{dx} \right|_{x=-2l+k-2}, \quad (22)$$

which have been evaluated numerically and listed in Table I for $p = 4$ as an example.

TABLE I
CONNECTION COEFFICIENTS FOR THE INTERPOLATING BASIS.

	$p=4$			
l	$a_{\phi\phi}$	$a_{\phi\psi}$	$a_{\psi\phi}$	$a_{\psi\psi}$
-8		-0.0000018		
-7		0.0000397		
-6	-0.0000109	-0.0004033		0.0000109
-5	-0.0008309	0.0030967		0.0008309
-4	0.0086543	-0.0144141	-0.0000017	-0.0086543
-3	-0.0419957	0.0394952	0.0044481	0.0426846
-2	0.1560101	-0.0637289	0.3839979	-0.2903309
-1	-1.3110341	0.0533016	0.0	-1.8610039
0	1.3110341	0.0	-0.3839979	1.8610039
1	-0.1560101	-0.0533016	-0.0044481	0.2903309
2	0.0419957	0.0637289	0.0000017	-0.0426846
3	-0.0086543	-0.0394952		0.0086543
4	0.0008309	0.0144141		-0.0008309
5	0.0000109	-0.0030967		-0.0000109
6		0.0004033		
7		-0.0000397		
8		0.0000018		

III. Analysis of Dielectric Waveguides

A dielectric waveguide shown in Fig. 2 is tested with the proposed scheme using the Deslauriers-Dubuc interpolating function of $p = 4$ (DD_4) as well as the conventional FDTD method [14]. The waveguide is surrounded by perfect electric conductor walls. Three analysis conditions were tested

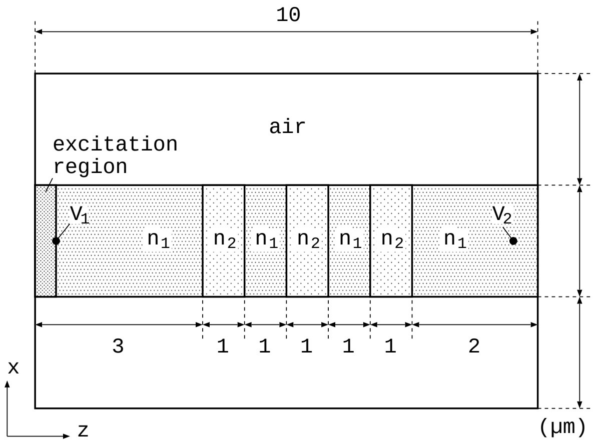

Fig. 2. Dielectric waveguide tested. Refractive indexes $n_1 = 3$ and $n_2 = 2.828$.

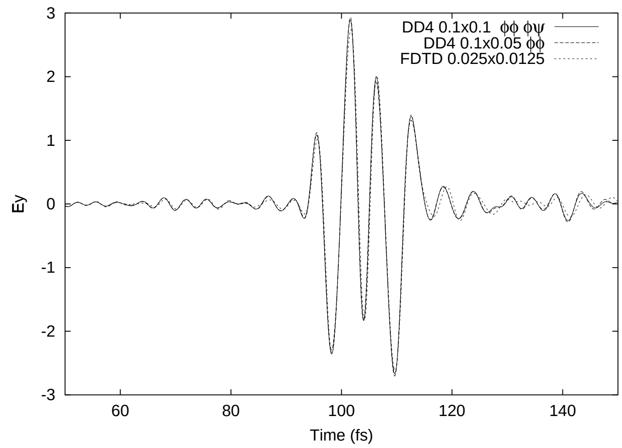

Fig. 3. Comparison of the time series data at output point V_2.

as listed in Table II. In scheme (i), the wavelet term $\phi\psi$ was added only at the core region of the waveguide where the main pulse propagates and finer resolution is desired. This is equivalent to the sub-gridding scheme in the conventional space discrete methods. When FDTD is used, the discretization must be significantly smaller than the present method, resulting in excessive computational expenditure for practical problems. The resulting time series data are plotted

TABLE II

ANALYSIS CONDITIONS FOR THE DIELECTRIC WAVEGUIDE.

	Scheme	Δx (μm)	Δz (μm)	Time steps	CPU time (s)
(i)	DD$_4$, $\phi\phi$ and $\phi\psi$	0.1	0.1	3180	3m1s
(ii)	DD$_4$, $\phi\phi$ only	0.1	0.05	5028	5m4s
(iii)	FDTD	0.025	0.0125	4022	11m21s

in Fig. 3. Good agreement has been achieved between the present scheme and the conventional FDTD, while computational resources are significantly reduced; the present scheme required only less than one third of CPU time and about one fifth of memory of those of FDTD.

IV. CONCLUSION

The biorthogonal interpolating wavelets have been applied to electromagnetic field analysis through the time-domain wavelet-Galerkin scheme. The algorithm has been applied to the analysis of two-dimensional dielectric waveguides that have typical dimension of optical waveguides. The interpolation bases associated with their duals of linear combination of Diracs yield schemes of arbitrary orders of regularity while saving the numerical overhead of field reconstruction process. The proposed scheme is particularly efficient for electrically-large problems such as optical waveguides that have been difficult to solve with the conventional FDTD method due to the excessive computational expenditure.

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